Hyperbolic Celtic Knot Patterns

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Abstract

Centuries ago, Celtic knot patterns were used to decorate religious texts. Celtic knots are formed by weaving bands in an alternating over-and-under pattern. Originally, these were finite patterns on the Euclidean plane. Recently such patterns have also been drawn on spheres, thus utilizing a second of the three “classical geometries”. We complete the process by exhibiting Celtic knot patterns in hyperbolic geometry, the third classical geometry. Our methods lead to a unified framework for discussing knot patterns in each of the classical geometries. Because of the precision and many calculations required to construct hyperbolic patterns, it is natural to generate such patterns by computer. Thus, the patterns we show are created by using computers, mathematics, and aesthetic considerations.

Introduction

In about the 6th century Irish monks started using what we now call Celtic knot patterns as ornamentation for religious texts. The monks also created spiral patterns, key patterns, zoomorphic patterns, and decorated lettering, but we will only consider knot patterns. Figure 1 shows a simple example of a knot pattern. The

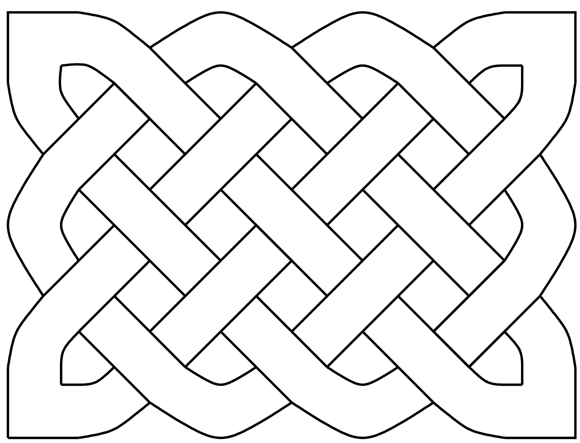

Figure 1: A simple Celtic knot pattern

use of this kind of decoration went out of style in about the 10th century, and the methods for creating such patterns were lost as well. Subsequently, people who wanted to make Celtic knot patterns had to copy existing patterns. That is, until the early 1950’s when George Bain invented a method for creating such patterns [1].

In the late 1950's, the Dutch artist M. C. Escher became the first person to create hyperbolic art in his four *Circle Limit* patterns. The pattern of interlocking rings near the edge of his last woodcut *Snakes* (Catalog Number 448 of [6]) also exhibits hyperbolic symmetry. The goal of this paper is to take a first step toward combining Celtic knot art and hyperbolic geometry. Thus Celtic knot patterns will have been drawn on each of the three *classical geometries*: Euclidean, spherical (or elliptical), and hyperbolic geometry. Celtic knot patterns have also been drawn on convex polyhedra, which are very closely related to spherical patterns.

We will begin with a brief review of Celtic knots and hyperbolic geometry, followed by a discussion of regular tessellations, which form the basis for our hyperbolic Celtic knot patterns. Finally, we will develop a theory of such patterns, showing some samples, and indicate directions of future work.

Celtic Knot Patterns

Celtic knot patterns were used in the British Isles to decorate stonework and religious texts from the sixth through the tenth centuries. The methods used by monks to create such patterns have been lost. However, in 1951, George Bain published a method to create such patterns which he discovered after years of studying those ancient patterns. Later, his son, Iain Bain, published a simplified algorithm for making knot patterns in 1986 [2]. It is Iain Bain's method, as explained by Andrew Glassner [3], that we will discuss here.

The simplest knot patterns can be constructed from a rectangular grid of squares as shown in Figure 2. The set of vertices of this grid, thought of as a graph, form the starting point for Iain Bain's construction and is called the *primary grid* by Glassner. The center points of the squares form the vertices of another rectangular grid of squares as shown in Figure 3 (with its edges extended to the boundary of the primary grid). This is the *secondary grid*. The *tertiary grid*, shown in Figure 4, is formed by the union of the primary grid and the secondary grid. Thus, the tertiary grid is a grid of squares of half the edge width of the squares in the primary and secondary grids.

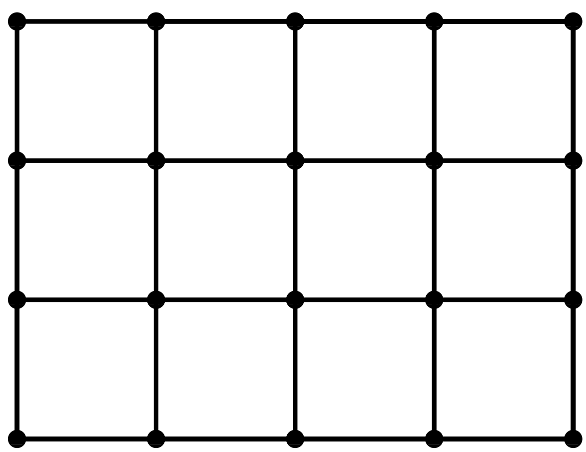

Figure 2: The primary grid for a Celtic knot construction.

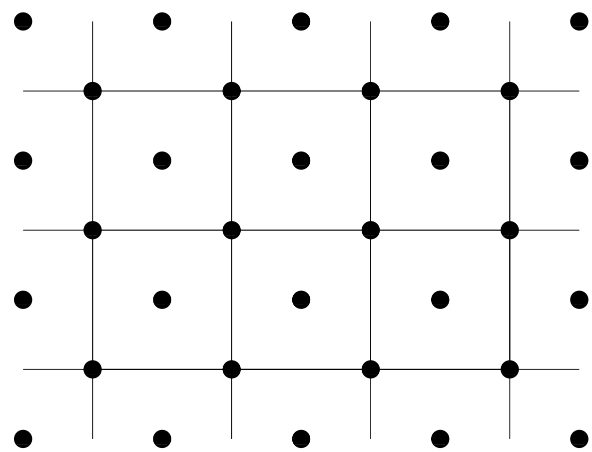

Figure 3: The secondary grid for a Celtic knot construction.

Diagonal lines are drawn in each of the interior small squares of the tertiary grid using the lower-left to upper-right diagonal for the upper-left interior square, and then drawing the rest of the diagonals in an alternating pattern of lower-left to upper-right and upper-left to lower-right diagonals as in Figure 5. These diagonals will form what is called the *internal weaving.* The internal weaving in this example will be a plait, seen in the interior pattern of Figure 1.

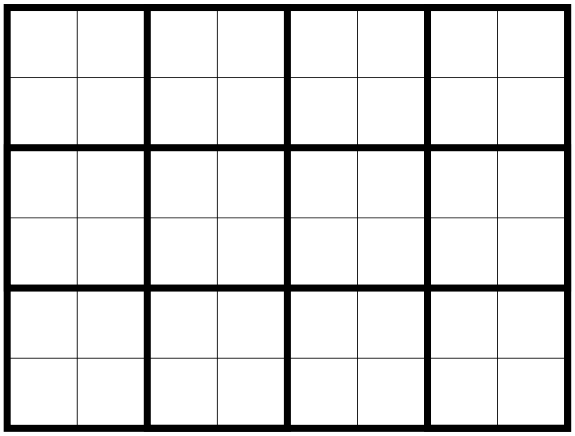

Figure 4: The tertiary grid — the union of the primary grid (heavy lines) and the secondary grid (light lines).

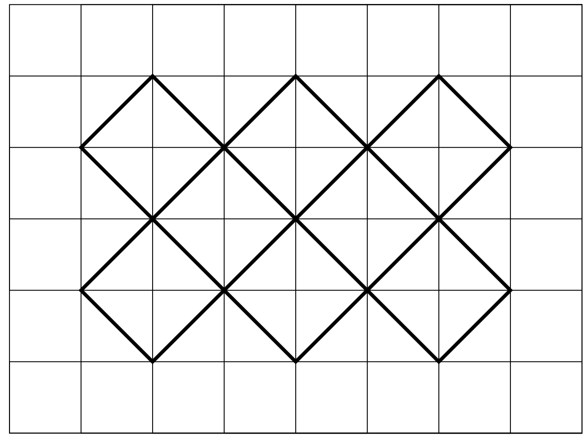

Figure 5: The diagonals in the interior of the tertiary grid that form the internal weaving.

Diagonals are also placed in the small edge squares of the tertiary grid in the same pattern but only going halfway to the outer edge as shown in Figure 6. These diagonals form the *external weaving*, which will connect the ends of the internal weaving. Next, at each of the tertiary grid points where four diagonals meet, form two paths by connecting the lower-left to the upper-right diagonal, and connecting the upper-left to the lower-right diagonal. Following one of the paths, let it go alternately above and below the paths it crosses. This can be done in a consistent way by using one kind of crossing on each row of crossing points and then using the other kind of crossing on the next row, as in Figure 7.

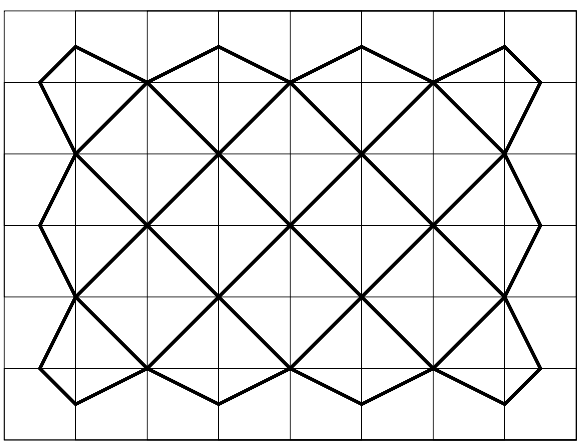

Figure 6: The outer diagonals that form the external weaving (in addition to the internal weaving).

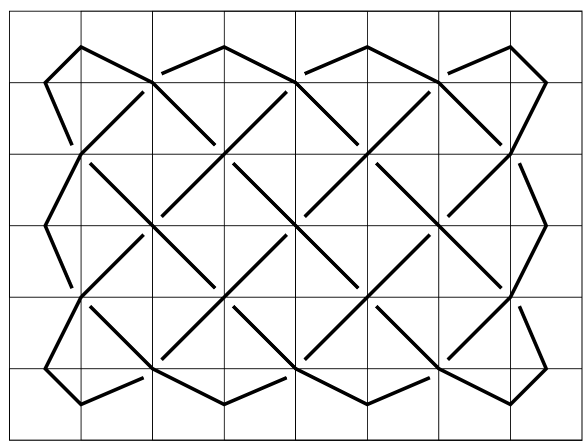

Figure 7: The over-and-under specification of the path.

Using knot theory terminology, the over-and-under pattern formed by the diagonals is the *regular projection* (onto the plane) of a knot (a circle embedded in 3-space); it is regular because only two strands cross at a point. A "multi-knot" formed by more than one circle in 3-space is called a *link*. There will be only one path if the numbers of rows and columns of vertices in the primary grid are relatively prime. Most Celtic knot patterns are the regular projections of knots: there is only one path. The path or paths serve as the centerlines of the bands of the final pattern, which is formed by thickening the paths to form the bands. The bands are usually thickened to a width equal to the distance between them (so the standard band thickness and the space between them are both equal to half the length of the diagonal of a primary grid square). Figure 1 shows the final result for the example we have been studying. Some Celtic patterns use wider bands with almost no space between them. Other patterns use thin doubled bands that follow the edges of the standard thickness bands.

More General Patterns

The interior weaving of the pattern described above is very regular — it amounts to a tiling by alternate rows of left- and right-handed crossings. These crossings are enclosed in kite-shaped tiles, actually square tiles tilted at 45 degrees, as shown in Figure 8 (except that the center tile contains a non-crossing, as discussed below). The top and bottom vertices of these kite tiles are both primary grid vertices or both secondary grid vertices; the left and right vertices of the kites are vertices of the other grid.

To obtain more general patterns, one can replace some or all of the "crossing" tiles by either of the *avoiding* tiles shown in Figure 9. We call those tiles *vertical* or *horizontal* avoiding tiles because their paths avoid either the vertical or horizontal axis of their kite-shaped tile. Each such replacement may increase or decrease the number of loops in a link by one, or it may leave the number unchanged. If, after replacing all the crossing tiles by avoiding tiles, there is only one loop, it is called a *snake* by Glassner [4].

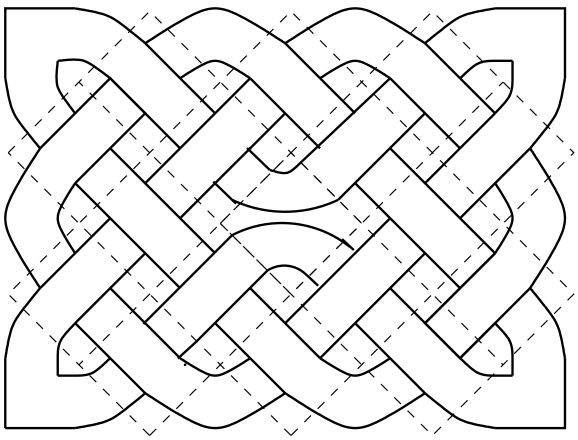

Figure 8: The kite-shaped tiles underlying a Celtic knot pattern.

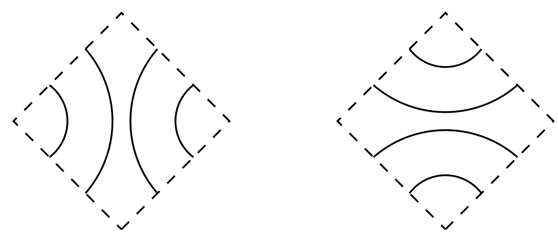

Figure 9: The vertical (left) and horizontal (right) avoiding tiles.

One can also create a non-rectangular pattern by arranging the crossing and avoiding tiles in any simply-connected way and then joining the ends of the bands around the perimeter. One method for creating such patterns by hand involves lightly drawing the primary and secondary grids and then drawing more darkly some of the edges of either grid, with the rule that no dark edges may cross. These dark edges are *barrier* edges that the band cannot cross. In Figure 8 there is a horizontal barrier edge (not shown) connecting the left and right (secondary grid) vertices of the center kite. Glassner [3] and Christian Mercat [7] describe their versions of this method. Barrier edges are called *breaklines* by Glassner, and *longitudinal* and *transverse walls* by Mercat depending on whether they are edges of the primary or secondary grid.

With the goal of generalizing these techniques to the hyperbolic plane, we next discuss hyperbolic geometry, repeating patterns, and regular tessellations, which will form the basis for hyperbolic Celtic knot patterns.

Among the classical geometries, the Euclidean plane, the sphere, and the hyperbolic plane, the latter is certainly the least familiar. This is probably due to the fact that there is no smooth distance-preserving embedding of the hyperbolic plane into ordinary 3-space, as there is for the sphere (and the Euclidean plane). However, there are *models* of hyperbolic geometry in the Euclidean plane, which must therefore distort distance.

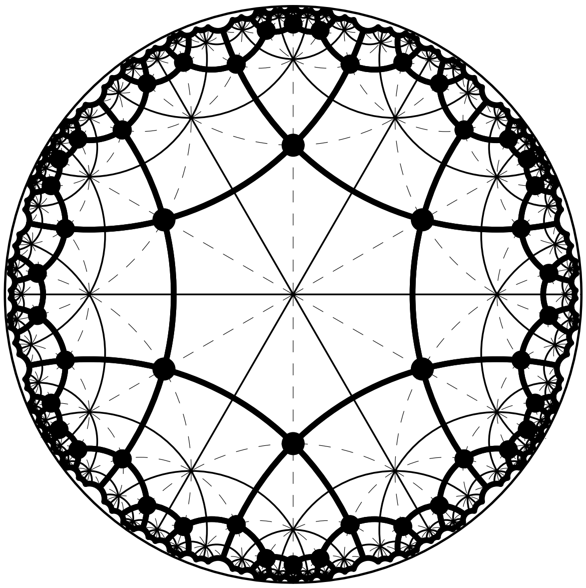

Figure 10: The regular tessellation $\{6, 4\}$ in heavy lines with dots at its vertices, its dual tessellation $\{4, 6\}$ in light lines, and common radii of the 6-gons and 4-gons in dashed lines.

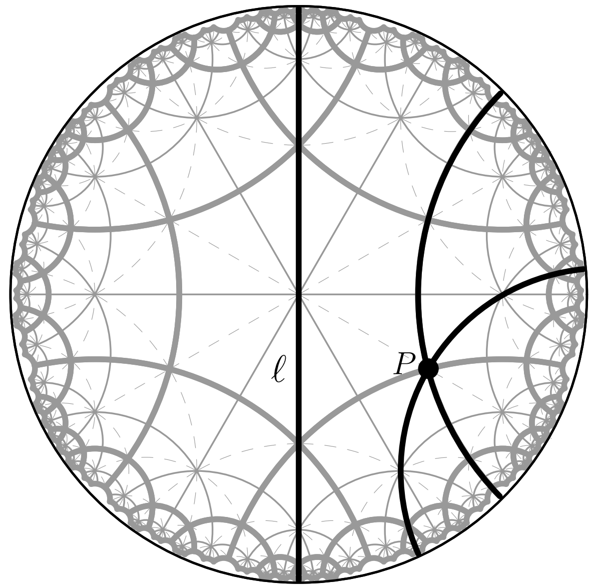

Figure 11: An example of the hyperbolic parallel property: a line ℓ, a point P not on ℓ, and two lines through P not meeting ℓ.

One of these models is the *Poincaré circle model*, which has two useful properties: (1) it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure) — consequently a transformed object has roughly the same shape as the original, and (2) it lies entirely within a *bounding circle* in the Euclidean plane — allowing an entire hyperbolic pattern to be displayed. In this model, the hyperbolic points are the interior points of the bounding circle and the hyperbolic lines are interior circular arcs perpendicular to the bounding circle, including diameters. For example, all the arcs are hyperbolic lines in Figure 10.

By definition, (plane) hyperbolic geometry satisfies all the axioms of (plane) Euclidean geometry except the Euclidean parallel axiom, which is replaced by its negation. Figure 11 shows an example of this hyperbolic parallel property: there is a line, ℓ, in Figure 10 (the vertical diameter), a point, P, not on it, and more than one line through P that does not intersect ℓ.

Because distances must be distorted in any model, equal hyperbolic distances in the Poincaré model are represented by ever smaller Euclidean distances toward the edge of the bounding circle (which is an infinite hyperbolic distance from its center). All the curvilinear hexagons (actually regular hyperbolic hexagons) in Figure 10 are the same hyperbolic size, even thought they are represented by different Euclidean sizes.

A *repeating pattern* in any of the classical geometries is a pattern made up of congruent copies of a basic subpattern or *motif*. The motif for the pattern of Figure 10 is a curvilinear right triangle with a dashed hypotenuse and thick and thin lines for legs. Also, we assume that a repeating pattern fills up its respective plane. It is useful that hyperbolic patterns repeat in order to show their true hyperbolic nature.

An important kind of repeating pattern in any of the classical geometries is the *regular tessellation* by regular p-sided polygons, or p-*gons*, meeting q at a vertex; it is denoted by the Schläfli symbol $\{p,q\}$. We need $(p-2)(q-2) > 4$ to obtain a hyperbolic tessellation; if $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$, one obtains tessellations of the Euclidean plane and the sphere, respectively. Figure 10 shows the hyperbolic tessellation $\{6,4\}$ in heavy lines with a 6-gon centered in the bounding circle (the center of the bounding circle is not a special point in the Poincaré model, it just appears so to our Euclidean eyes). Figure 10 also shows the hyperbolic tessellation $\{4,6\}$ in light lines with one of its vertices centered in the bounding circle. The dashed lines in Figure 10 do not form a regular tessellation, but when $p = q$ the analogous dashed lines form the regular tessellation $\{4,p\}$.

If we assume for simplicity that $p \geq 3$ and $q \geq 3$, there are five solutions to the "spherical" inequality $(p-2)(q-2) < 4$: $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{4,3\}$, and $\{5,3\}$. These tessellations may be obtained by "blowing up" the Platonic solids: the regular tetrahedron, the octahedron, the icosahedron, the cube, and the dodecahedron, respectively, onto their circumscribing spheres. In the Euclidean case, there are three solutions to the equality $(p-2)(q-2) = 4$: $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$, the tessellations of the plane by equilateral triangles, squares, and regular hexagons. There are infinitely many solutions to the hyperbolic inequality $(p-2)(q-2) > 4$. This is summarized in Table 1 below.

q \ p	1	2	3	4	5	6	7	8	9	10	11	···
⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
11			*	*	*	*	*	*	*	*	*	...
10			*	*	*	*	*	*	*	*	*	...
9			*	*	*	*	*	*	*	*	*	...
8			*	*	*	*	*	*	*	*	*	...
7			*	*	*	*	*	*	*	*	*	...
6			□	*	*	*	*	*	*	*	*	...
5			○	*	*	*	*	*	*	*	*	...
4			○	□	*	*	*	*	*	*	*	...
3			○	○	○	□	*	*	*	*	*	...
2												
1												

□ - Euclidean tessellations

○ - spherical tessellations

* - hyperbolic tessellations

Table 1. The relationship between the values of p and q, and the geometry of the tessellation $\{p,q\}$.

For each tessellation $\{p,q\}$, its *dual tessellation* is $\{q,p\}$, whose vertices are at the centers of the p-gons of $\{p,q\}$ and whose edges are perpendicular bisectors of the edges of $\{p,q\}$. Figure 10 shows the tessellation $\{6,4\}$ in heavy lines and its dual tessellation $\{4,6\}$ in thin lines. Of course the dual of the dual of a regular tessellation is just the original tessellation. If $p = q$, the tessellation is self-dual: $\{3,3\}$ is the spherical version of the regular tetrahedron, $\{4,4\}$ is familiar Euclidean tiling by squares, and $\{5,5\}$, $\{6,6\}$, $\{7,7\}, \ldots$ are hyperbolic.

This completes our discussion of hyperbolic geometry, repeating patterns, and regular tessellations. Next, we use these concepts to develop a theory of hyperbolic Celtic knot patterns, which is actually valid in all three of the classical geometries.

As we saw above, the method for creating knot patterns that was developed by Iain Bain and others is based on the regular tessellation of the Euclidean plane by squares. We extend that method to one based on any regular tessellation of one of the classical geometries. The tessellation $\{p, q\}$ itself serves as the primary grid, its dual, $\{q, p\}$, defines the secondary grid, and their union is the tertiary grid. In Figure 10, where $p = 6$ and $q = 4$, the primary grid is shown in heavy lines and the secondary grid in thin (solid) lines. The dashed lines in Figure 10 define a tessellation by kite-shaped tiles — rhombuses with vertex angles of $2\pi/p$, $2\pi/q$, $2\pi/p$, and $2\pi/q$ (with vertices alternately at the centers and vertices of p-gons of the tessellation $\{p, q\}$). If one starts with the Euclidean $\{4, 4\}$ tessellation, the rhombuses are actually squares tilted at a 45-degree angle, as shown in Figure 8.

Celtic knot patterns have two characteristics: (1) no more than two bands cross at a point, and (2) any one band goes alternately over and under other bands that it crosses. Such a pattern can be obtained if all the rhombuses are filled in only with left crossing tiles or only with right crossing tiles. Looking at a rhombus from a primary grid vertex, if the nearest band coming from the right is on top, it is a *right crossing tile*, otherwise it is a *left crossing tile*; both kinds are shown in Figure 12 (the rhombuses shown are the ones to the right of the center of the bounding circle in Figure 10). Figure 13 shows a complete pattern composed of right crossing tiles based on the $\{4, 5\}$ tessellation. Such a Celtic pattern is called a *regular weaving* or *plait*. The central pattern in Figure 1 is another example — of the standard Euclidean weaving.

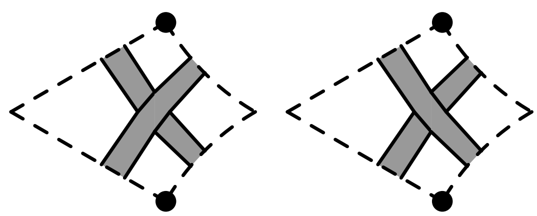

Figure 12: A left crossing tile (left) and a right crossing tile (right), with dots at the primary grid vertices.

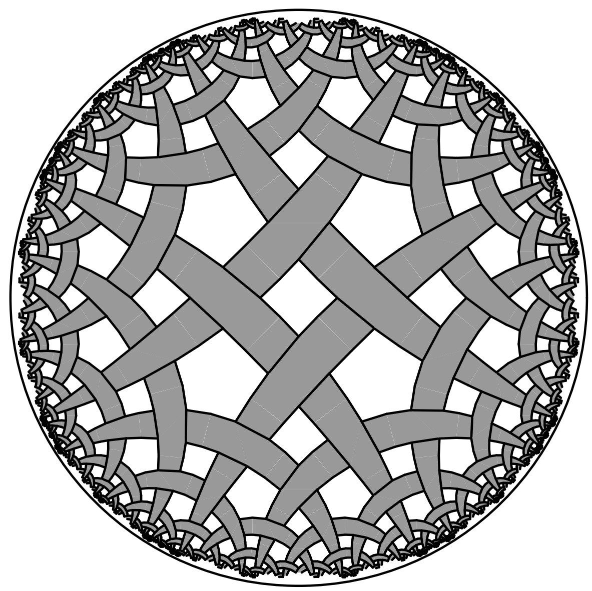

Figure 13: A regular weaving or plait based on the $\{4, 5\}$ tessellation.

The bands of a regular weaving based on the tessellation $\{p, q\}$ follow the edges of the *uniform tessellation* $(p.q.p.q)$ (also called *Archimedean* or *semiregular* tilings by some authors). The edges of $(p.q.p.q)$ are formed by connecting the midpoints of adjacent edges of the p-gons of $\{p, q\}$. Those midpoints serve as the vertices of $(p.q.p.q)$, each of which is surrounded by a p-gon, a q-gon, a p-gon, and a q-gon (which explains the notation). Figure 14 shows the uniform tessellation $(4, 5, 4, 5)$ underlying the regular weaving knot pattern of Figure 14. Since p-gon edge midpoints are also q-gon edge midpoints in the dual tessellation $\{q, p\}$, a regular tessellation and its dual produce the same regular weaving — which is not surprising since p and q play symmetrical roles in $(p.q.p.q)$.

There are three regular spherical weavings, which are based on the self-dual tessellation $\{3,3\}$, and on the two pairs of duals, $\{3,4\}$ and $\{4,3\}$, and $\{3,5\}$ and $\{5,3\}$. The weaving based on $\{3,3\}$ traces the edges of the “uniform” tessellation (3.3.3.3), which is actually the regular tessellation $\{3,4\}$, the blown-up version of the octahedron. There is a band in each of three mutually perpendicular planes through the center of the sphere containing the $\{3,3\}$. These three bands are linked, forming Borromean rings. Glassner shows such a weaving based on a cube rather than an octahedron (Figure 10a of [5]). The octahedron is the intersection of the tetrahedron $\{3,3\}$ and its dual, which together form the stella octangula. In fact, the regular weaving based on any self-dual tessellation $\{p,p\}$ traces the edges of the regular tessellation $\{p,4\}$. The weaving based on the pair $\{3,4\}$ and $\{4,3\}$ traces the edges of uniform tessellation (3.4.3.4), which is the spherical version of the cuboctahedron. Last, the weaving based on the pair $\{3,5\}$ and $\{5,3\}$ traces the edges of uniform tessellation (3.5.3.5), which is the spherical version of the icosadodecahedron. Glassner shows a version of this weaving in Figure 18 of [5].

There are only two regular Euclidean weavings, which are based on the self-dual tessellation $\{4,4\}$, and on the dual pair $\{3,6\}$ and $\{6,3\}$. The weaving based on $\{4,4\}$ is just the standard Euclidean weaving seen in the center of Figure 1, which is the basis for most Celtic knot patterns. The weaving based on $\{3,6\}$ and $\{6,3\}$, with its triangular and hexagonal holes, is sometimes seen in the caning for the seats of chairs.

There are infinitely many regular hyperbolic weavings— based either on the self-dual tessellations $\{p,p\}$ for $p \geq 5$, or on the dual pairs $\{p,q\}$ and $\{q,p\}$, where $p \neq q$ and $(p-2)(q-2) > 4$. Figures 15 and 13 show the weavings based on the self-dual $\{5,5\}$ tessellation and on the pair $\{4,5\}$ and $\{5,4\}$, respectively.

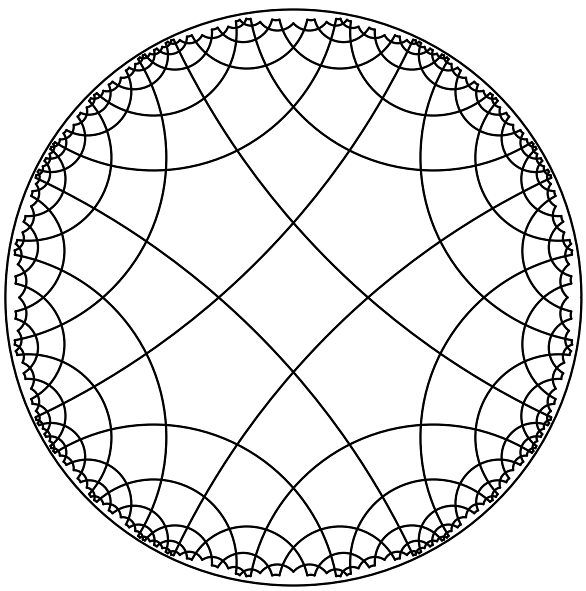

Figure 14: The uniform tessellation (4.5.4.5) underlying the regular weaving of Figure 13.

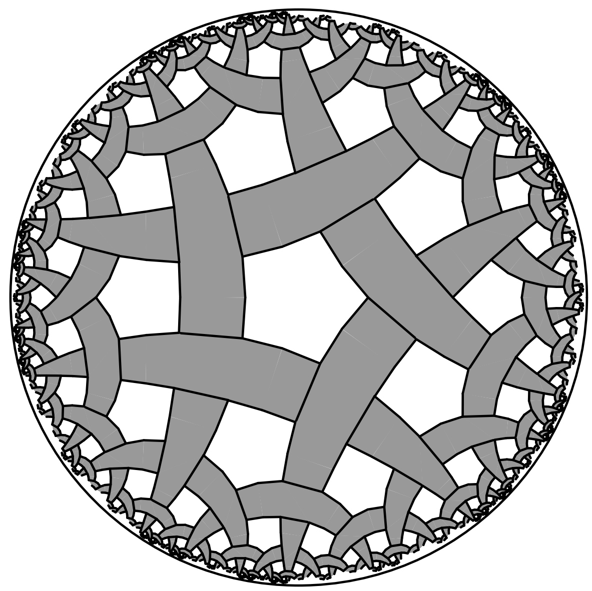

Figure 15: The regular weaving based on the tessellation $\{5,5\}$.

More general Celtic knot patterns may be obtained by replacing some of the crossing tiles of a regular weaving with *avoiding tiles*. Figure 16 shows the two kinds of avoiding tiles, which are distinguished by the diagonal of the tile rhombus that their paths avoid (as in Figure 12, the rhombuses shown are the ones to the right of the center of the bounding circle in Figure 10); Figure 9 shows the avoiding tiles for the standard Eulidean weaving (based on $\{4,4\}$). One of the diagonals of each rhombus is an edge from the underlying tessellation $\{p,q\}$, and the other diagonal is an edge from the dual tessellation $\{q,p\}$. Figure 17 shows a pattern of alternating right crossing tiles and p-gon edge avoiding tiles. Figure 18 shows a pattern

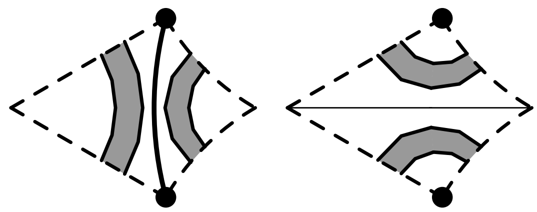

Figure 16: A p-gon edge avoiding tile (left) and a q-gon edge avoiding tile (right), with dots at the primary grid vertices.

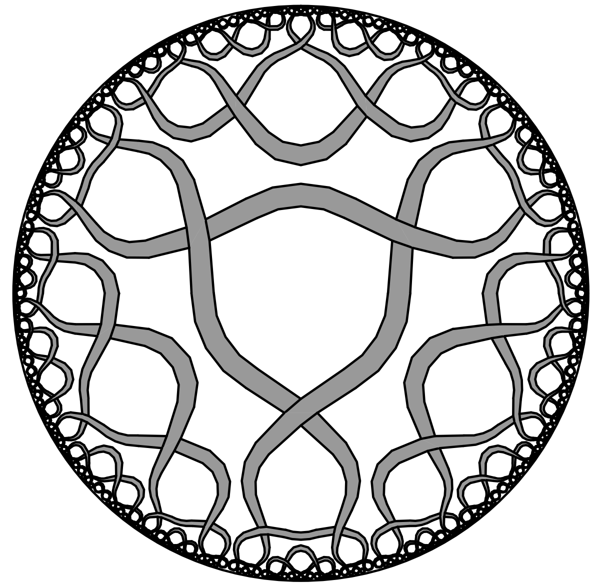

Figure 17: A Celtic knot pattern of right crossing tiles and p-gon edge avoiding tiles.

of alternating right crossing tiles and q-gon edge avoiding tiles.

One of the rules of Celtic knot patterns is that paths cannot avoid both diagonals of a rhombus (there would be no way to connect the ends the paths coming into that tile). Thus, if we want to construct Celtic knot patterns from rhombus tiles, our collection of basic tiles is complete, consisting of the two kinds of crossing tiles and the two kinds of avoiding tiles.

It is possible to further generalize the methods above to apply to non-rhombic quadrilateral tiles. For any quadrilateral, there are only four ways to connect ends of bands coming into it across each of its four sides: the two kinds of crossing configurations and the two kinds of avoiding configurations. Glassner has used non-rhombic quadrilaterals to construct several of his patterns in [5]. As an example, if we have a pattern of triangles upon which we would like to draw a knot pattern, we could first subdivide the triangles into three quadrilaterals by connecting the triangle's center to the midpoints of its sides.

We will apply this method to the construction of what we call *Celtic ring patterns* — rings interlocked in the over-and-under pattern characteristic of Celtic knots. We start by subdividing the p-gons of the tessellation $\{p, q\}$ into p isosceles triangles with angles $2\pi/p$, π/q, and π/q, as shown in the central p-gon in Figure 19 (where $p = 6$ and $q = 4$). Then we subdivide each triangle into three quadrilaterals (shown for one of the isosceles triangles in Figure 19). Finally, we place a crossing tile in each of the quadrilaterals, producing the final ring pattern of Figure 19. Note that the crossing is pushed as far as possible toward one vertex of the quadrilateral. Figure 19 shows the pattern of interlocking rings that Escher used near the edge of his last woodcut, *Snakes* (Catalog Number 448 of [6]).

This finishes our discussion of the theory of hyperbolic Celtic knot patterns and the methods for creating them. Of course, the theory and methods also apply to each of the three classical geometries as well. In the final section, we indicate directions of future work.

Future Work

We have presented a theory of Celtic knot patterns and methods for creating such patterns in each of

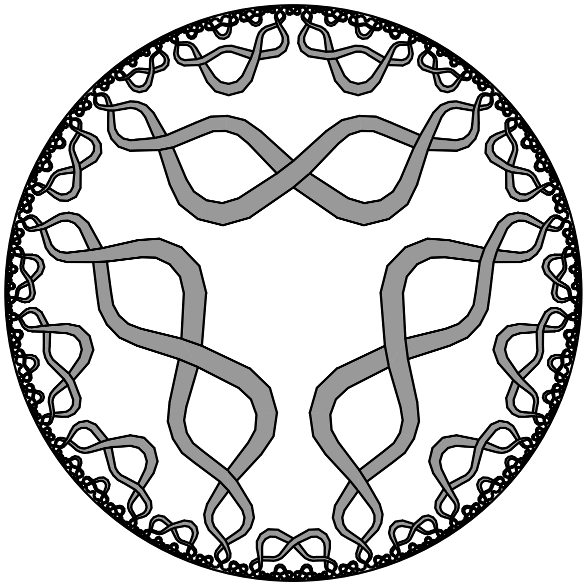

Figure 18: A Celtic knot pattern of right crossing tiles and q-gon edge avoiding tiles.

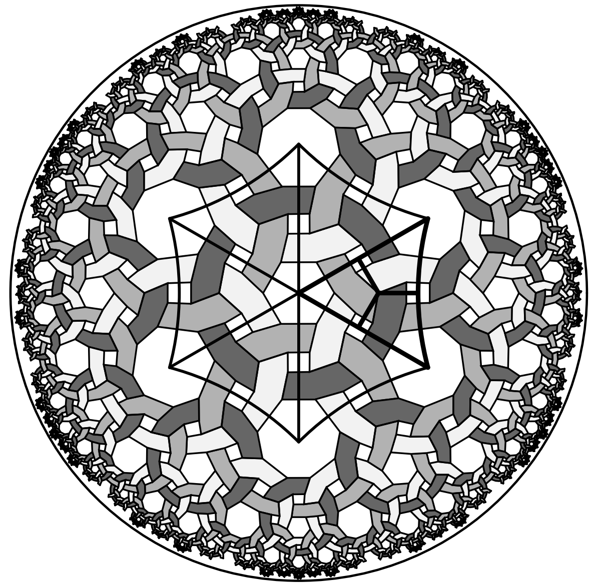

Figure 19: An interlocking "Celtic ring" pattern showing part of the underlying $\{6,4\}$ and some of the triangles used in the construction, with one of them subdivided into three quadrilaterals.

the three classical geometries. Some natural directions of future work include extensions to hyperbolic knot patterns not based on regular tessellations, and the creation of hyperbolic versions of other kinds of Celtic patterns, such as key patterns, spiral patterns and zoomorphics.

Acknowledgment

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